Arithmetic progression

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Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that

$$|a_m + a_n - a_{m+n}| \leq \frac{1}{m+n} \text{ for all } m,n \in \mathbb{N}.$$

Prove that sequence is AP.

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By replacing (n,m) in inequality $|a_{m+n} - a_m - a_n| \leq \dfrac{1}{m+n}$ with $(n+k-1,1)$, for $\forall n,k \in \mathbb{N}$ we obtain $|a_{n+k} - a_{n+k-1} - a_1| \leq \dfrac{1}{n+k} \Leftrightarrow a_1 - \dfrac{1}{n+k} \leq a_{n+k} - a_{n+k-1} \leq a_1 + \dfrac{1}{n+k}$.

Then for any $p \in \mathbb{N}$ we have

$$\sum_{k=1}^{p}\left(a_1 - \frac{1}{n+k}\right) \leq \sum_{k=1}^{p}(a_{n+k} - a_{n+k-1}) \leq \sum_{k=1}^{p}\left(a_1 + \frac{1}{n+k}\right) \Leftrightarrow$$

(1) $\quad pa_1 - \sum_{k=1}^{p}\dfrac{1}{n+k} \leq a_{n+p} - a_n \leq pa_1 + \sum_{k=1}^{p}\dfrac{1}{n+k}$.

Since for any $n,p \in \mathbb{N}$ holds inequality

(2) $\quad |a_{n+p} - a_n - a_p| \leq \dfrac{1}{n+p} \Leftrightarrow a_p - \dfrac{1}{n+p} \leq a_{n+p} - a_n \leq a_p + \dfrac{1}{n+p}$

then (**1**) and (**2**) implies

$$a_p - \frac{1}{n+p} \leq pa_1 + \sum_{k=1}^{p}\frac{1}{n+k} \Leftrightarrow a_p - pa_1 \leq \frac{1}{n+p} + \sum_{k=1}^{p}\frac{1}{n+k}$$

and $pa_1 - \sum_{k=1}^{p}\dfrac{1}{n+k} \leq a_p + \dfrac{1}{n+p} \Leftrightarrow -\dfrac{1}{n+p} - \sum_{k=1}^{p}\dfrac{1}{n+k} \leq a_p - pa_1$.

Thus, $|a_p - pa_1| \leq \dfrac{1}{n+p} + \sum_{k=1}^{p}\dfrac{1}{n+k}$ for any $n,p \in \mathbb{N}$ and then

$|a_p - pa_1| \leq \lim\limits_{n\to\infty}\left(\dfrac{1}{n+p} + \sum_{k=1}^{p}\dfrac{1}{n+k}\right) = 0$ implies

$$a_p - pa_1 = 0 \Leftrightarrow a_p = pa_1 \text{ for any } p \in \mathbb{N}.$$